

SHARP ESTIMATES OF THE POTENTIAL KERNEL FOR THE HARMONIC OSCILLATOR WITH APPLICATIONS

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ABSTRACT. We prove qualitatively sharp estimates of the potential kernel for the harmonic oscillator. These bounds are then used to show that the $L^p - L^q$ estimates of the associated potential operator obtained recently by Bongioanni and Torrea [2] are in fact sharp.

1. INTRODUCTION

The study of the potential theory for the d -dimensional harmonic oscillator

$$\mathcal{H} = -\Delta + \|x\|^2,$$

has recently been initiated by Bongioanni and Torrea [2]. The multi-dimensional Hermite functions h_k are eigenfunctions of \mathcal{H} and we have $\mathcal{H}h_k = (2|k| + d)h_k$. The operator \mathcal{H} has a natural self-adjoint extension, here still denoted by \mathcal{H} , whose spectral decomposition is given by the h_k .

The integral kernel $G_t(x, y)$ of the Hermite semigroup $\{\exp(-t\mathcal{H}) : t > 0\}$ is known explicitly to be (see [7] for this symmetric variant of the formula)

$$\begin{aligned} G_t(x, y) &= \sum_{n=0}^{\infty} e^{-(2n+d)t} \sum_{|k|=n} h_k(x) h_k(y) \\ &= (2\pi \sinh(2t))^{-d/2} \exp \left(-\frac{1}{4} \left[\tanh(t) \|x + y\|^2 + \coth(t) \|x - y\|^2 \right] \right). \end{aligned}$$

Given $\sigma > 0$, consider the negative power $\mathcal{H}^{-\sigma}$, which is a contraction on $L^2(\mathbb{R}^d)$. It is easily seen that $\mathcal{H}^{-\sigma}$ coincides in $L^2(\mathbb{R}^d)$ with the *potential operator*

$$(1) \quad \mathcal{I}^\sigma f(x) = \int_{\mathbb{R}^d} \mathcal{K}^\sigma(x, y) f(y) dy,$$

where the *potential kernel* is given by

$$(2) \quad \mathcal{K}^\sigma(x, y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty G_t(x, y) t^{\sigma-1} dt.$$

Note that all the spaces $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, are contained in the natural domain of \mathcal{I}^σ consisting of those functions f for which the integral in (1) converges x -a.e., see [5, Section 2].

The main result of the paper, Theorem 2.4 below, provides qualitatively sharp estimates of the potential kernel (2). As an application of this result, we prove sharpness of the $L^p - L^q$ estimates for the potential operator (1) obtained recently by Bongioanni and Torrea [2, Theorem 8], see Theorem 3.1.

Recall that an operator T defined on $L^p(\mathbb{R}^d)$ for some $1 \leq p \leq \infty$, with values in the space of measurable functions on \mathbb{R}^d , is said to be of weak type (p, q) , $1 \leq q < \infty$, provided that

$$(3) \quad |\{x \in \mathbb{R}^d : |Tf(x)| > \lambda\}| \leq C \left(\|f\|_p / \lambda \right)^q,$$

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with $C > 0$ independent of $f \in L^p(\mathbb{R}^d)$ and $\lambda > 0$. The restricted weak type (p, q) of T means that (3) holds for $f = \chi_E$, where E is any measurable subset of \mathbb{R}^d of finite measure. By definition, weak type (p, ∞) coincides with strong type (p, ∞) , i.e. the estimate $\|Tf\|_\infty \leq C\|f\|_p$, $f \in L^p(\mathbb{R}^d)$. In terms of Lorentz spaces, the weak type (p, q) is equivalent to the boundedness from $L^p(\mathbb{R}^d)$ to $L^{q,\infty}(\mathbb{R}^d)$, and the restricted weak type (p, q) is characterized by the boundedness from $L^{p,1}(\mathbb{R}^d)$ to $L^{q,\infty}(\mathbb{R}^d)$, see [1, Chapter 4, Section 4]. Strong type (p, q) means of course the L^p - L^q boundedness.

The notation $X \lesssim Y$ will be used to indicate that $X \leq CY$ with a positive constant C independent of significant quantities; we shall write $X \simeq Y$ when simultaneously $X \lesssim Y$ and $Y \lesssim X$. We will also use the notation $X \simeq Y \exp(-cZ)$ to indicate that there exist positive constants C, c_1 and c_2 , independent of significant quantities, such that

$$C^{-1}Y \exp(-c_1Z) \leq X \leq CY \exp(-c_2Z).$$

Further, in a number of places, we will use natural and self-explanatory generalizations of the \simeq relation, for instance in connection with certain integrals involving exponential factors. In such cases the exact meaning will be clear from the context. By convention, \simeq is understood as \simeq whenever there are no exponential factors involved.

We write \log^+ for the positive part of the logarithm, and \vee, \wedge for the operations of taking maximum and minimum, respectively.

2. ESTIMATES OF THE POTENTIAL KERNEL

We begin with two technical results describing the behavior of the integrals

$$\begin{aligned} I_A(T) &= \int_T^\infty t^A \exp(-t) dt, & T > 0, \\ J_A(T, S) &= \int_T^S t^A \exp(-t) dt, & 0 < T < S < \infty. \end{aligned}$$

Notice that $I_A(T)$ dominates $J_A(T, S)$. The lemma below is a refinement of [5, Lemma 2.1], see also [6, Lemma 1.1].

Lemma 2.1. *Let $A \in \mathbb{R}$ and $\gamma > 0$ be fixed. Then*

$$(4) \quad I_A(\gamma T) \simeq T^A \exp(-\gamma T), \quad T \geq 1,$$

and for $0 < T < 1$

$$I_A(\gamma T) \simeq \begin{cases} T^{A+1}, & A < -1 \\ \log(2/T), & A = -1 \\ 1, & A > -1 \end{cases}.$$

Proof. We assume that $\gamma = 1$. From the proof it will be clear that the estimates are true for any $\gamma > 0$. The case $0 < T < 1$ was treated in the proof of [5, Lemma 2.1], so we consider $T \geq 1$ and focus on showing (4). The lower bound in (4) is straightforward, we have

$$I_A(T) > \int_T^{2T} t^A e^{-t} dt \gtrsim T^A \int_T^{2T} e^{-t} dt = T^A (e^{-T} - e^{-2T}) \gtrsim T^A e^{-T}, \quad T \geq 1.$$

It remains to prove the upper bound,

$$(5) \quad \int_T^\infty t^A e^{-t} dt \lesssim T^A e^{-T}, \quad T \geq 1,$$

and here we assume that $A > 0$, since for $A \leq 0$ we have $t^A \leq T^A$, $t > T \geq 1$, and the conclusion is trivial. Choosing T_A such that for $T \geq T_A$ one has

$$\int_{2T}^{\infty} t^A e^{-t} dt \leq \frac{1}{2} \int_T^{\infty} t^A e^{-t} dt,$$

we can write

$$\int_T^{\infty} t^A e^{-t} dt \leq \int_T^{2T} t^A e^{-t} dt + \int_{2T}^{\infty} t^A e^{-t} dt \leq C T^A e^{-T} + \frac{1}{2} \int_T^{\infty} t^A e^{-t} dt, \quad T \geq T_A.$$

This implies (5) for $T \geq T_A$ and consequently for all $T \geq 1$. \square

Lemma 2.2. *Let $A \in \mathbb{R}$ and $\gamma > 0$ be fixed. Then for $0 < T < S \leq 2T$ we have*

$$(6) \quad T^A(S - T) \exp(-2\gamma T) \lesssim J_A(\gamma T, \gamma S) \lesssim T^A(S - T) \exp(-\gamma T),$$

while for $S > 2T > 0$ we have $J_A(\gamma T, \gamma S) \simeq I_A(\gamma T)$ when $S \geq 2$, and

$$J_A(\gamma T, \gamma S) \simeq \begin{cases} T^{A+1}, & A < -1 \\ \log(S/T), & A = -1 \\ S^{A+1}, & A > -1 \end{cases}$$

when $0 < S < 2$.

Proof. As in the proof of Lemma 2.1, it is enough to deal with the case $\gamma = 1$. The bounds for $T < S \leq 2T$ follow since then $\int_T^S t^A e^{-t} dt \simeq T^A \int_T^S e^{-t} dt$ and

$$(S - T)e^{-2T} \leq \int_T^S e^{-t} dt \leq (S - T)e^{-T}.$$

Assume now that $S > 2T$. Clearly, $J_A(T, S) < I_A(T)$. On the other hand, if $T \geq 1$ then

$$J_A(T, S) > \int_T^{2T} t^A e^{-t} dt \gtrsim T^A \int_T^{2T} e^{-t} dt \gtrsim T^A e^{-T} \gtrsim I_A(T),$$

the last estimate being a consequence of (4). When $0 < T < 1$, we distinguish two subcases. If $S \geq 2$, then again $J_A(T, S) \gtrsim \int_T^2 t^A dt \gtrsim I_A(T)$. If $2T < S < 2$, then $J_A(T, S) \simeq \int_T^S t^A dt$, and evaluating the last integral we arrive at the claimed bounds for $J_A(T, S)$. \square

We note that (4) and (6) may be written slightly less precisely as

$$\begin{aligned} I_A(\gamma T) &\simeq \exp(-cT), & T \geq 1, \\ J_A(\gamma T, \gamma S) &\simeq T^A(S - T) \exp(-cT), & 0 < T < S \leq 2T, \end{aligned}$$

respectively. This fact will be used in the sequel without further mention.

We now apply Lemmas 2.1 and 2.2 to prove qualitatively sharp estimates of the integral

$$E_A(T, S) = \int_0^1 t^A \exp(-Tt^{-1} - St) dt, \quad 0 < T, S < \infty.$$

The following result provides, in particular, a refinement and generalization of [3, Lemma 2.4].

Lemma 2.3. *Let $A \in \mathbb{R}$ be fixed. Then*

$$E_A(T, S) \simeq \exp\left(-c\sqrt{T(T \vee S)}\right) \times \begin{cases} T^{A+1}, & A < -1 \\ 1 + \log^+ \frac{1}{T(T \vee S)}, & A = -1 \\ (S \vee 1)^{-A-1}, & A > -1 \end{cases},$$

uniformly in $T, S > 0$.

Proof. We first estimate $E_A(T, S)$ in terms of the integrals I_A and J_A . For $0 < S \leq 2T$ we have

$$E_A(T, S) \simeq \int_0^1 t^A \exp(-cTt^{-1}) dt \simeq T^{A+1} \int_{cT}^\infty u^{-A-2} e^{-u} du = T^{A+1} I_{-A-2}(cT),$$

where the second relation follows by the change of variable $t = cT/u$. When $S > 2T$ we change the variable $t = u\sqrt{T/S}$ and get

$$E_A(T, S) = \left(\frac{T}{S}\right)^{(A+1)/2} \int_0^{\sqrt{S/T}} u^A \exp(-\sqrt{TS}(u + u^{-1})) du \equiv \mathcal{J}_1 + \mathcal{J}_2,$$

where \mathcal{J}_1 and \mathcal{J}_2 come from splitting the integration over the intervals $(0, 1)$ and $(1, \sqrt{S/T})$, respectively. Then

$$\begin{aligned} \mathcal{J}_1 &\simeq \left(\frac{T}{S}\right)^{(A+1)/2} \int_0^1 u^A \exp(-c\sqrt{TS}u^{-1}) du \simeq T^{A+1} \int_{c\sqrt{TS}}^\infty z^{-A-2} e^{-z} dz \\ &= T^{A+1} I_{-A-2}(c\sqrt{TS}) \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}_2 &\simeq \left(\frac{T}{S}\right)^{(A+1)/2} \int_1^{\sqrt{S/T}} u^A \exp(-c\sqrt{TS}u) du \simeq S^{-A-1} \int_{c\sqrt{TS}}^{cS} z^A e^{-z} dz \\ &= S^{-A-1} J_A(c\sqrt{TS}, cS). \end{aligned}$$

Summing up, we have

$$E_A(T, S) \simeq T^{A+1} I_{-A-2}(c\sqrt{T(T \vee S)}) + \chi_{\{S > 2T\}} S^{-A-1} J_A(c\sqrt{TS}, cS),$$

uniformly in $S, T > 0$. In the next step we describe the behavior of the two terms here by means of Lemmas 2.1 and 2.2.

From Lemma 2.1 it follows that

$$T^{A+1} I_{-A-2}(c\sqrt{T(T \vee S)}) \simeq T^{A+1} \exp(-c\sqrt{T(T \vee S)}), \quad T(T \vee S) \geq 1,$$

(here, and also in analogous places below, c on the left-hand side should be understood as a *given* constant) and

$$T^{A+1} I_{-A-2}(c\sqrt{T(T \vee S)}) \simeq \begin{cases} T^{A+1}, & A < -1 \\ \log(\frac{4}{T(T \vee S)}), & A = -1 \\ (\frac{T}{T \vee S})^{(A+1)/2}, & A > -1 \end{cases}, \quad T(T \vee S) \leq 1.$$

The term $S^{-A-1} J_A(c\sqrt{TS}, cS)$ comes into play when $S > 2T$, and in this case we use Lemma 2.2 to write the bounds

$$S^{-A-1} J_A(c\sqrt{TS}, cS) \simeq \chi_{\{S \geq 2\}} \Phi_1 + \chi_{\{S < 2\}} \Phi_2,$$

where

$$\Phi_1 = S^{-A-1} I_A(c\sqrt{TS}), \quad \Phi_2 = \begin{cases} (T/S)^{(A+1)/2}, & A < -1 \\ \log(\frac{S}{T}), & A = -1 \\ 1, & A > -1 \end{cases}.$$

By Lemma 2.1,

$$\begin{aligned} \Phi_1 &\simeq S^{-A-1} \exp(-c\sqrt{TS}), \quad TS \geq 1, \\ \Phi_1 &\simeq \begin{cases} (T/S)^{(A+1)/2}, & A < -1 \\ \log(\frac{4}{TS}), & A = -1 \\ S^{-A-1}, & A > -1 \end{cases}, \quad TS \leq 1. \end{aligned}$$

To proceed, it is convenient to consider each of the cases $A < -1$, $A = -1$, and $A > -1$ separately.

If $A < -1$, then

$$E_A(T, S) \simeq \chi_{\{2 > S > 2T\}} \left(\frac{T}{S} \right)^{(A+1)/2} + \begin{cases} T^{A+1} \exp(-c\sqrt{T(T \vee S)}), & T(T \vee S) \geq 1 \\ T^{A+1}, & T(T \vee S) < 1 \end{cases} \\ + \chi_{\{S > 2T\}} \chi_{\{S \geq 2\}} \begin{cases} T^{A+1} \exp(-c\sqrt{TS}), & TS \geq 1 \\ \left(\frac{T}{S} \right)^{(A+1)/2}, & TS < 1 \end{cases}.$$

Here the first and third terms are insignificant in comparison to the second one. In case of the third summand, this is because $A < -1$ and $\left(\frac{T}{S} \right)^{(A+1)/2} < T^{A+1}$ for $TS < 1$. A similar argument is used for the first one. The required estimates of $E_A(T, S)$ follow.

If $A = -1$, then

$$E_{-1}(T, S) \simeq \chi_{\{2 > S > 2T\}} \log \frac{S}{T} + \begin{cases} \exp(-c\sqrt{T(T \vee S)}), & T(T \vee S) \geq 1 \\ \log \left(\frac{4}{T(T \vee S)} \right), & T(T \vee S) < 1 \end{cases} \\ + \chi_{\{S > 2T\}} \chi_{\{S \geq 2\}} \begin{cases} \exp(-c\sqrt{TS}), & TS \geq 1 \\ \log \left(\frac{4}{TS} \right), & TS < 1 \end{cases}.$$

Similarly as in the case of $A < -1$, here also the first and third terms are insignificant in comparison to the second one. This is clear for the third summand, and for the first one this is because $\log \frac{S}{T} < \log \left(\frac{4}{TS} \right)$ when $S < 2$. Thus the desired bounds of $E_{-1}(T, S)$ also follow.

Finally, we consider the case $A > -1$, which is less direct than the previous two. We have

$$E_A(T, S) \simeq \chi_{\{2 > S > 2T\}} + \begin{cases} T^{A+1} \exp(-c\sqrt{T(T \vee S)}), & T(T \vee S) \geq 1 \\ \left(\frac{T}{T \vee S} \right)^{(A+1)/2}, & T(T \vee S) < 1 \end{cases} \\ + \chi_{\{S > 2T\}} \chi_{\{S \geq 2\}} \begin{cases} T^{A+1} \exp(-c\sqrt{TS}), & TS \geq 1 \\ S^{-A-1}, & TS < 1 \end{cases}.$$

Observe that here the relation \simeq remains valid if the sum of the first and the third terms is replaced by the comparable (in the sense of \simeq) expression

$$\chi_{\{S > 2T\}} \begin{cases} T^{A+1} \exp(-c\sqrt{TS}), & TS \geq 1 \\ (S \vee 1)^{-A-1}, & TS < 1 \end{cases}.$$

Taking into account that $T^{A+1} \exp(-c\sqrt{TS}) \simeq S^{-A-1} \exp(-c\sqrt{TS})$ for $TS \geq 1$, we conclude that

$$E_A(T, S) \simeq \begin{cases} (T \vee S)^{-A-1} \exp(-c\sqrt{T(T \vee S)}), & T(T \vee S) \geq 1 \\ \left(\frac{T}{T \vee S} \right)^{(A+1)/2}, & T(T \vee S) < 1 \end{cases} \\ + \chi_{\{S > 2T\}} \begin{cases} S^{-A-1} \exp(-c\sqrt{TS}), & TS \geq 1 \\ (S \vee 1)^{-A-1}, & TS < 1 \end{cases}.$$

Now, if $T \geq S$ and $T(T \vee S) = T^2 < 1$, then $\left(\frac{T}{T \vee S} \right)^{1/2} = 1 \simeq 1/(S \vee 1)$, while for $T < S$ and $T(T \vee S) = TS < 1$, we have $\left(\frac{T}{T \vee S} \right)^{1/2} = \left(\frac{T}{S} \right)^{1/2} < 1/(S \vee 1)$. Therefore,

$$E_A(T, S) \simeq \begin{cases} (T \vee S)^{-A-1} \exp(-c\sqrt{T(T \vee S)}), & T(T \vee S) \geq 1 \\ (S \vee 1)^{-A-1}, & T(T \vee S) < 1 \end{cases}.$$

We claim that this implies

$$E_A(T, S) \simeq (S \vee 1)^{-A-1} \exp(-c\sqrt{T(T \vee S)}),$$

which are precisely the required estimates.

To justify the claim, it is enough to recall that $A > -1$ and observe that if $T \geq S$ and $T(T \vee S) = T^2 \geq 1$, then

$$\begin{aligned} (T \vee S)^{-A-1} \exp(-c\sqrt{T(T \vee S)}) &= T^{-A-1} \exp(-cT) \simeq (T \vee 1)^{-A-1} \exp(-cT) \\ &\simeq (S \vee 1)^{-A-1} \exp(-cT), \end{aligned}$$

while if $T < S$ and $T(T \vee S) = TS \geq 1$ (this forces $S > 1$), then

$$(T \vee S)^{-A-1} \exp(-c\sqrt{T(T \vee S)}) = S^{-A-1} \exp(-c\sqrt{TS}) \simeq (S \vee 1)^{-A-1} \exp(-c\sqrt{TS}).$$

The proof is finished. \square

We are now in a position to prove qualitatively sharp estimates of the potential kernel.

Theorem 2.4. *For $\sigma > 0$ we have*

$$\mathcal{K}^\sigma(x, y) \simeq \exp(-c\|x - y\|(\|x\| + \|y\|)) \times \begin{cases} \|x - y\|^{2\sigma-d}, & \sigma < d/2 \\ 1 + \log^+ \frac{1}{\|x - y\|(\|x\| + \|y\|)}, & \sigma = d/2 \\ (1 + \|x + y\|)^{d-2\sigma}, & \sigma > d/2 \end{cases},$$

uniformly in $x, y \in \mathbb{R}^d$.

Proof. We decompose

$$\Gamma(\sigma)\mathcal{K}^\sigma(x, y) = \int_0^1 G_t(x, y) t^{\sigma-1} dt + \int_1^\infty G_t(x, y) t^{\sigma-1} dt \equiv \mathcal{J}_0^\sigma(x, y) + \mathcal{J}_\infty^\sigma(x, y).$$

For $0 < t < 1$ we have $\tanh t \simeq t$, $\coth t \simeq t^{-1}$, $\sinh 2t \simeq t$, and therefore

$$\mathcal{J}_0^\sigma(x, y) \simeq E_{\sigma-d/2-1}(c\|x - y\|^2, c\|x + y\|^2).$$

This combined with Lemma 2.3 shows that the estimates from the statement hold with $\mathcal{K}^\sigma(x, y)$ replaced by $\mathcal{J}_0^\sigma(x, y)$. Further, taking into account that $\tanh t \simeq 1 \simeq \coth t$ for $t > 1$, we see that

$$\mathcal{J}_\infty^\sigma(x, y) \simeq \exp(-c(\|x\|^2 + \|y\|^2)).$$

Thus $\mathcal{J}_0^\sigma(x, y)$ dominates $\mathcal{J}_\infty^\sigma(x, y)$ in the above decomposition, in the sense that

$$\mathcal{J}_\infty^\sigma(x, y) \lesssim E_{\sigma-d/2-1}(c\|x - y\|^2, c\|x + y\|^2)$$

for a sufficiently small constant $c > 0$. The conclusion follows. \square

3. SHARPNESS OF THE L^p - L^q BOUNDEDNESS OF THE POTENTIAL OPERATOR

Given $0 < \sigma < d/2$, define the region

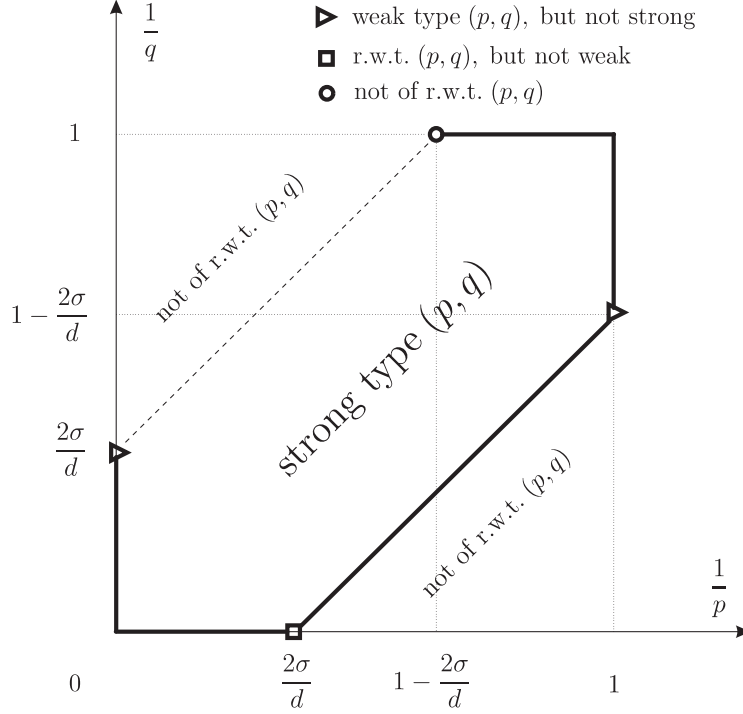
$$\begin{aligned} R = & \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) : 0 \leq \frac{1}{p} \leq 1 \text{ and } 0 \vee \left(\frac{1}{p} - \frac{2\sigma}{d} \right) \leq \frac{1}{q} \leq 1 \wedge \left(\frac{1}{p} + \frac{2\sigma}{d} \right) \right\} \\ & \setminus \left(\left\{ \left(\frac{1}{p}, \frac{1}{q} \right) : 0 \leq \frac{1}{p} \leq 1 - \frac{2\sigma}{d} \text{ and } \frac{1}{q} = \frac{1}{p} + \frac{2\sigma}{d} \right\} \cup \left\{ \left(\frac{2\sigma}{d}, 0 \right), \left(1, 1 - \frac{2\sigma}{d} \right) \right\} \right) \end{aligned}$$

contained in the unit $(\frac{1}{p}, \frac{1}{q})$ -square $[0, 1]^2$, see Figure 1.

The following result enhances [2, Theorem 8], see also [5, Theorem 2.3].

Theorem 3.1. *Let $d \geq 1$, $0 < \sigma < d/2$ and $1 \leq p, q \leq \infty$. Then $\mathcal{I}^\sigma : L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d)$ boundedly if and only if $(\frac{1}{p}, \frac{1}{q})$ lies in the region R .*

On the other hand, \mathcal{I}^σ is not even of restricted weak type (p, q) when $(\frac{1}{p}, \frac{1}{q}) \notin R$, except for the two cases $(\frac{1}{p}, \frac{1}{q}) = (0, \frac{2\sigma}{d})$ and $(\frac{1}{p}, \frac{1}{q}) = (1, 1 - \frac{2\sigma}{d})$ in which weak type (p, q) inequalities hold, and the singular case $(\frac{1}{p}, \frac{1}{q}) = (\frac{2\sigma}{d}, 0)$, in which the restricted weak type is true, but weak type fails.

FIGURE 1. Mapping properties of \mathcal{I}^σ for $0 < \sigma < d/2$.

Before giving the proof we take the opportunity to present a short argument showing [2, (21) and (41)], the result we will apply in a moment.

Lemma 3.2. *Given $\sigma > 0$,*

$$\|\mathcal{K}^\sigma(x, \cdot)\|_1 \simeq (1 \vee \|x\|)^{-2\sigma}, \quad x \in \mathbb{R}^d.$$

Proof. Using the identity (see [7, Proposition 3.3])

$$\exp(-t\mathcal{H})\mathbf{1}(x) = \int_{\mathbb{R}^d} G_t(x, y) dy = (\cosh 2t)^{-d/2} \exp\left(-\frac{1}{2} \tanh(2t)\|x\|^2\right), \quad x \in \mathbb{R}^d,$$

we may write

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{K}^\sigma(x, y) dy &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_{\mathbb{R}^d} G_t(x, y) dy t^{\sigma-1} dt \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty (\cosh 2t)^{-d/2} \exp\left(-\frac{1}{2} \tanh(2t)\|x\|^2\right) t^{\sigma-1} dt. \end{aligned}$$

Here we split the integration to the intervals $(0, 1)$ and $(1, \infty)$ and denote the resulting integrals by \mathcal{J}_0 and \mathcal{J}_∞ , respectively. Then, uniformly in $x \in \mathbb{R}^d$,

$$\mathcal{J}_0 \simeq \int_0^1 \exp(-ct\|x\|^2) t^{\sigma-1} dt = \|x\|^{-2\sigma} \int_0^{\|x\|^2} e^{-cs} s^{\sigma-1} ds \simeq \|x\|^{-2\sigma} (\|x\|^{2\sigma} \wedge 1)$$

and

$$\mathcal{J}_\infty \simeq \int_1^\infty e^{-td} \exp(-c\|x\|^2) t^{\sigma-1} dt = C_{d,\sigma} \exp(-c\|x\|^2).$$

The conclusion follows. \square

Proof of Theorem 3.1. We first focus on strong type inequalities. Then, in view of [2, Theorem 8], what remains to prove are the following two items.

- (a) \mathcal{I}^σ is not $L^p - L^q$ bounded for $\frac{2\sigma}{d} < \frac{1}{p} < 1$ and $0 < \frac{1}{q} < \frac{1}{p} - \frac{2\sigma}{d}$.
- (b) \mathcal{I}^σ is not $L^p - L^q$ bounded for $0 < \frac{1}{p} < 1 - \frac{2\sigma}{d}$ and $\frac{1}{p} + \frac{2\sigma}{d} \leq \frac{1}{q} < 1$.

To justify (a), we fix p and q satisfying the assumed conditions and define

$$f(y) = \chi_{\{\|y\| < 1\}} \|y\|^{-2\sigma-d/q}.$$

This function is in $L^p(\mathbb{R}^d)$ since $-(2\sigma+d/q)p+d > 0$. However, $\mathcal{I}^\sigma f \notin L^q(\mathbb{R}^d)$. Indeed, considering x such that $\|x\| < 1$ and using the lower bound from Theorem 2.4 we get

$$\mathcal{I}^\sigma f(x) \gtrsim \int_{\|y\| < \|x\|/2} \|x-y\|^{2\sigma-d} \|y\|^{-2\sigma-d/q} dy \gtrsim \|x\|^{2\sigma-d} \int_{\|y\| < \|x\|/2} \|y\|^{-2\sigma-d/q} dy = C \|x\|^{-d/q},$$

and the function $x \mapsto \chi_{\{\|x\| < 1\}} \|x\|^{-d/q}$ does not belong to $L^q(\mathbb{R}^d)$.

Proving (b) we may assume that $(\frac{1}{p}, \frac{1}{q})$ lies on the critical segment $\frac{1}{q} = \frac{1}{p} + \frac{2\sigma}{d}$, $0 < \frac{1}{p} < 1 - \frac{2\sigma}{d}$. The case when $\frac{1}{q} > \frac{1}{p} + \frac{2\sigma}{d}$ is implicitly contained in what follows, and can also be covered by a simplified counterexample not involving the logarithmic factor. Define

$$f(y) = \chi_{\{\|y\| > e\}} \|y\|^{-d/p} (\log \|y\|)^{-1/p-2\sigma/d}.$$

We have

$$\int_{\mathbb{R}^d} |f(y)|^p dy = C_d \int_e^\infty r^{-1} (\log r)^{-1-2\sigma p/d} dr < \infty,$$

so $f \in L^p(\mathbb{R}^d)$. We claim that $\mathcal{I}^\sigma f \notin L^q(\mathbb{R}^d)$. Assuming that $\|x\| > 2e$ and using the lower bound from Theorem 2.4 we write

$$\begin{aligned} \mathcal{I}^\sigma f(x) &\gtrsim \int_{\|x\|/2 < \|y\| < \|x\|} \|x-y\|^{2\sigma-d} \exp(-c\|x-y\|(\|x\| + \|y\|)) \|y\|^{-d/p} (\log \|y\|)^{-1/p-2\sigma/d} dy \\ &\gtrsim \|x\|^{-d/p} (\log \|x\|)^{-1/p-2\sigma/d} \int_{\|x\|/2 < \|y\| < \|x\|} \|x-y\|^{2\sigma-d} \exp(-2c\|x-y\|\|y\|) dy. \end{aligned}$$

As we shall see in a moment, the last integral is comparable with $\|x\|^{-2\sigma}$. Thus

$$\mathcal{I}^\sigma f(x) \gtrsim \|x\|^{-d/p-2\sigma} (\log \|x\|)^{-1/p-2\sigma/d} = \|x\|^{-d/q} (\log \|x\|)^{-1/q}, \quad \|x\| > 2e,$$

and the claim follows.

It remains to analyze the last integral, which we denote by \mathcal{J} . Changing the variable $y = x - z/\|x\|$ we get

$$\mathcal{J} = \|x\|^{-2\sigma} \int_{D_x} \|z\|^{2\sigma-d} e^{-2c\|z\|} dz,$$

where the set of integration is $D_x = \{z \in \mathbb{R}^d : \|x\|^2/2 < \|x\|x\| - \|z\| < \|x\|^2\}$. We now observe that D_x contains the ball $B_x = \{x \in \mathbb{R}^d : \|x\|x\|/4 - \|z\| < \|x\|^2/4\}$. Indeed, if $z \in B_x$ then

$$\frac{\|x\|^2}{2} < \left\| \frac{\|x\|x\|}{4} - z \right\| - \left\| \frac{3}{4}x\|x\| \right\| \leq \|x\|x\| - \|z\| \leq \left\| \frac{\|x\|x\|}{4} - z \right\| + \left\| \frac{3}{4}x\|x\| \right\| < \|x\|^2.$$

Thus we have

$$\|x\|^{-2\sigma} \int_{B_x} \|z\|^{2\sigma-d} e^{-2c\|z\|} dz \leq \mathcal{J} \leq \|x\|^{-2\sigma} \int_{\mathbb{R}^d} \|z\|^{2\sigma-d} e^{-2c\|z\|} dz.$$

Clearly, the integral over \mathbb{R}^d here is finite. The integral over B_x depends on x only through $\|x\|$. Since the balls B_x are increasing in the sense of \subset when x is moved away from the origin along a fixed line passing through the origin, we see that the integral over B_x is an increasing function of $\|x\|$, which is positive and finite. We conclude that $\mathcal{J} \simeq \|x\|^{-2\sigma}$, $\|x\| > 2e$, as desired.

We pass to weak type and restricted weak type inequalities. Consider first the four ‘corners’ of the boundary of R for which the associated strong type inequalities fail. If $(\frac{1}{p}, \frac{1}{q}) = (1, 1 - \frac{2\sigma}{d})$,

then the weak type $(1, \frac{d}{d-2\sigma})$ holds by [5, Theorem 2.3]. Notice that this property can be expressed in terms of Lorentz spaces by saying that \mathcal{I}^σ is bounded from $L^1(\mathbb{R}^d)$ to $L^{d/(d-2\sigma), \infty}(\mathbb{R}^d)$. Then $(\mathcal{I}^\sigma)^*$ (the adjoint operator in the Banach space sense) maps boundedly $(L^{d/(d-2\sigma), \infty}(\mathbb{R}^d))^*$ into $(L^1(\mathbb{R}^d))^* = L^\infty(\mathbb{R}^d)$. Further, the associate space of $L^{d/(d-2\sigma), \infty}(\mathbb{R}^d)$ in the sense of [1, Chapter 1, Definition 2.3] is $L^{d/(2\sigma), 1}(\mathbb{R}^d)$ (cf. [1, Chapter 4, Theorem 4.7]), and by [1, Chapter 1, Theorem 2.9] it can be regarded as a subspace of the dual of $L^{d/(d-2\sigma), \infty}(\mathbb{R}^d)$. Since $(\mathcal{I}^\sigma)^* = \mathcal{I}^\sigma$ by symmetry of the kernel, we infer that \mathcal{I}^σ is of restricted weak type $(\frac{d}{2\sigma}, \infty)$. On the other hand, weak type $(\frac{d}{2\sigma}, \infty)$ coincides, by definition, with the strong type, so \mathcal{I}^σ is not of weak type $(\frac{d}{2\sigma}, \infty)$ in view of the strong type results we already know. This clarifies the situations related to the ‘corners’ $(1, 1 - \frac{2\sigma}{d})$ and $(\frac{2\sigma}{d}, 0)$.

Taking into account $(\frac{1}{p}, \frac{1}{q}) = (0, \frac{2\sigma}{d})$, we will show that \mathcal{I}^σ is of weak type $(\infty, \frac{d}{2\sigma})$. To do that, it is enough to verify the estimate

$$(7) \quad |\{x \in \mathbb{R}^d : |\mathcal{I}^\sigma f(x)| > \lambda\}| \lesssim \left(\frac{\|f\|_\infty}{\lambda} \right)^{d/(2\sigma)}, \quad \lambda > 0, \quad f \in L^\infty(\mathbb{R}^d).$$

But this is immediate in view of the bound, see Lemma 3.2,

$$\|\mathcal{K}^\sigma(x, \cdot)\|_1 \leq C\|x\|^{-2\sigma}, \quad x \in \mathbb{R}^d,$$

since then it follows that $|\mathcal{I}^\sigma f(x)| \leq C\|x\|^{-2\sigma}\|f\|_\infty$ and consequently

$$\{x \in \mathbb{R}^d : |\mathcal{I}^\sigma f(x)| > \lambda\} \subset \left\{x \in \mathbb{R}^d : \|x\| < \left(C \frac{\|f\|_\infty}{\lambda} \right)^{1/2\sigma} \right\}.$$

This inclusion leads directly to (7). Finally, in case of the remaining ‘corner’ $(\frac{1}{p}, \frac{1}{q}) = (1 - \frac{2\sigma}{d}, 1)$, we use an *au contraire* argument. If \mathcal{I}^σ were of restricted weak type $(\frac{d}{d-2\sigma}, 1)$, then by the Marcinkiewicz interpolation theorem for Lorentz spaces (cf. [1, Chapter 4, Theorem 4.13]) we would have strong type (p, q) for all points $(\frac{1}{p}, \frac{1}{q})$ lying inside the segment S with endpoints $(0, \frac{2\sigma}{d})$ and $(1 - \frac{2\sigma}{d}, 1)$, a contradiction with (b) above.

For all the points $(\frac{1}{p}, \frac{1}{q})$ lying inside the segment S , as well as those lying outside the closure of R , we apply again the interpolation argument with suitably chosen endpoints (see Figure 1). This shows that for those points $(\frac{1}{p}, \frac{1}{q})$ the operator \mathcal{I}^σ is not of restricted weak type (p, q) .

The proof is finished. \square

For completeness, we remark that in the case $\sigma > d/2$ the operator \mathcal{I}^σ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ for every $1 \leq p, q \leq \infty$, see [5, Theorem 2.3]. The behavior of \mathcal{I}^σ in the limiting case $\sigma = d/2$ is described by the theorem below. This result enhances [5, Theorem 2.3] when $\sigma = d/2$.

Theorem 3.3. *Let $d \geq 1$ and $1 \leq p, q \leq \infty$. Then $\mathcal{I}^{d/2}$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ except for $(p, q) = (\infty, 1)$ and $(p, q) = (1, \infty)$. Considering the two singular cases, we have:*

- (i) $\mathcal{I}^{d/2}$ is of weak type $(\infty, 1)$, but not of strong type $(\infty, 1)$;
- (ii) $\mathcal{I}^{d/2}$ is not of restricted weak type $(1, \infty)$.

Proof. The L^p - L^q boundedness is contained in [5, Theorem 2.3]. To show (i), we observe that the weak type $(\infty, 1)$ holds true since the proof of (7) covers also the case $\sigma = d/2$. The strong type $(\infty, 1)$ fails because $\mathcal{I}^{d/2}\mathbf{1} \notin L^1(\mathbb{R}^d)$, as easily seen by means of Lemma 3.2.

It remains to verify (ii). For $0 < \varepsilon < 1/e$, let $f_\varepsilon(x) = \chi_{\{\|x\| < \varepsilon\}}$. By the lower bound of Theorem 2.4 it follows that

$$\mathcal{I}^{d/2}f_\varepsilon(x) \gtrsim \int_{\|y\| < \varepsilon} \log \frac{1}{\|x - y\|(\|x\| + \|y\|)} dy, \quad \|x\| < 1/e,$$

uniformly in $\varepsilon < 1/e$. Therefore,

$$\|\mathcal{I}^{d/2} f_\varepsilon\|_\infty \gtrsim \int_{\|y\| < \varepsilon} -\log \|y\| dy = C_d \int_0^\varepsilon -r^{d-1} \log r dr \gtrsim \varepsilon^d \log \frac{1}{\varepsilon}, \quad 0 < \varepsilon < 1/e,$$

and we conclude that

$$\frac{\|\mathcal{I}^{d/2} f_\varepsilon\|_\infty}{\|f_\varepsilon\|_1} \gtrsim \log \frac{1}{\varepsilon}, \quad 0 < \varepsilon < 1/e.$$

Letting $\varepsilon \rightarrow 0^+$, we see that $\mathcal{I}^{d/2}$ is not of restricted weak type $(1, \infty)$. □

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